Last time: inner (dot) product $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + \dots + v_n w_n$

allows us to compute $\begin{cases} \text{lengths of vectors} & \|v\| = \sqrt{v \cdot v} \\ \text{angles between vectors} & \cos\theta = \dfrac{v \cdot w}{\|v\| \, \|w\|} \end{cases}$

Orthogonality $\begin{cases} \text{of vectors:} & v \perp w \iff v \cdot w = 0 \\ \text{of subspaces:} & V \perp W \iff v \cdot w = 0 \quad \forall v \in V, w \in W \end{cases}$

THM 21.6 if $V = \text{span}\{v_1, \dots, v_k\}$
$W = \text{span}\{w_1, \dots, w_\ell\}$, then $V \perp W \iff A^T B = 0$

where $A = (v_1 \dots v_k) \in \mathbb{R}^{n \times k}$
$B = (w_1 \dots w_\ell) \in \mathbb{R}^{n \times \ell}$

$V = Col(A)$
$W = Col(B)$

Proof: $A^T B = \begin{pmatrix} 2 \;\; 5 \;\; 1 \\ \vdots \\ v_k \end{pmatrix} \left(\begin{matrix} 3 \\ 0 \\ 7 \end{matrix} \middle| \cdots \middle| w_\ell \right)$ $\qquad v \cdot w = v^T w$

$= \begin{pmatrix} 2\cdot3+5\cdot0+1\cdot7 & \cdots \\ & \vdots \end{pmatrix} = \begin{pmatrix} v_1 \cdot w_1 & \cdots & v_1 \cdot w_\ell \\ \vdots & \ddots & \vdots \\ v_k \cdot w_1 & \cdots & v_k \cdot w_\ell \end{pmatrix}$

so $A^T B = 0 \iff v_i \cdot w_j = 0 \ \forall i,j \iff V \perp W$ □

Orthogonal complements: if $V \subseteq \mathbb{R}^n$ is a subspace, then

$V^\perp = \{ w \in \mathbb{R}^n \mid w \perp V \iff w \perp v \ \forall v \in V \}$ is also a subspace

Today: how to compute $V^\perp$? pick a basis $\{v_1, \dots, v_k\}$ of V and construct $A = (v_1 \dots v_k) \in \mathbb{R}^{n \times k} \iff V = \mathrm{Col}(A)$

THM 22.1: $\boxed{\mathrm{Col}(A)^\perp = \mathrm{Ker}(A^T)}$ $\forall$ matrix A

Proof: we will actually prove the "transposed" statement

$$\boxed{\mathrm{Row}(A)^\perp = \mathrm{Ker}(A)}$$

$v \in \mathrm{Row}(A)^\perp \overset{\text{THM 21.6}}{\iff} v \perp$ any vector in a basis of $\mathrm{Row}(A)$

$\iff v \perp$ the rows of A

$\iff Av = 0 \iff v \in \mathrm{Ker}(A)$

$A = \begin{pmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \end{pmatrix}$, $\quad Av = \begin{pmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \end{pmatrix} v = \begin{pmatrix} r_1^T \cdot v \\ r_2^T \cdot v \\ \vdots \end{pmatrix} = 0$

$$\begin{pmatrix} \vdots \\ r_k \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ r_k \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ r_k^T \cdot v \end{pmatrix}$$

$$\Updownarrow$$

$$v \perp \text{row}_1, \text{row}_2, \dots, \text{row}_k$$

Ex: $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$

2-dim

compute $V^\perp$

$4-2=2$-dim

$$V = \text{Col}(A), \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

Thm 22.1 says $V^\perp = \text{Ker}(A^T) = \text{Ker}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{pmatrix} \subset \mathbb{R}^4$

Gaussian elimination

$$= \text{Ker}\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+2z+3t \\ y-z-2t \\ 0 \end{pmatrix}$$

vanishes for

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -2z-3t \\ z+2t \\ z \\ t \end{pmatrix}$$

$$= \left\{ z \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \; t, z \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thm 22.2 $\quad V \subset \mathbb{R}^n$

THM 22.2 $\forall$ subspace $V \subset \mathbb{R}^n$

$$\dim V + \dim V^\perp = n$$

Proof: $V = \text{Row}(A)$, $A \in \mathbb{R}^{k \times n}$ where $k = \dim V$

$V^\perp = \text{Ker}(A)$ by Thm 22.1

rank nullity

Rank-nullity theorem: $\underbrace{\dim \text{Col}(A)}_{= \dim \text{Row}(A)} + \dim \text{Ker}(A) = n$

because both Col(A) and Row(A) have dim = # pivots

$\Rightarrow \dim V + \dim V^\perp = n$ $\square$

New topic: orthogonal bases

DEF 22.3: A set of vectors $\{v_1, \dots, v_k\}$ of $\mathbb{R}^n$ is called an orthogonal set if $v_i \perp v_j$ $\forall i \neq j$

PROP 22.4: any orthogonal set (that doesn't contain 0) is linearly indendent

$\Downarrow$

$|\text{the set}| = k \leq n$

$|\text{orthogonal set}| = k \leq n$

Proof: assume $\{v_1, \dots, v_k\}$ are linearly dependent

$\Downarrow$

$\exists\, i$ s.t. $v_i = c_1 v_1 + \dots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \dots + c_k v_k$

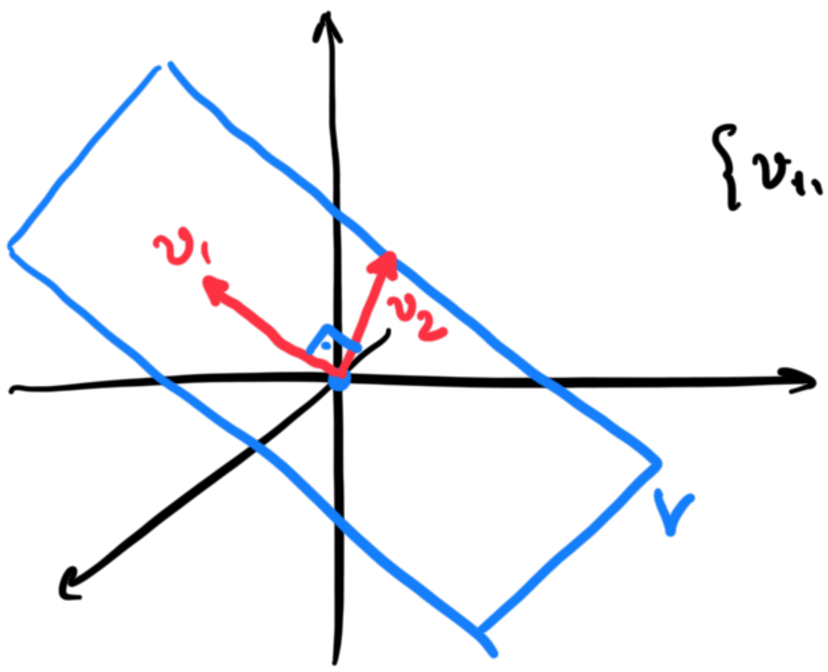
DOT this equality with v_i

$$0 \neq \|v_i\|^2 = v_i \cdot v_i = c_1 \underbrace{(v_1 \cdot v_i)}_{=0} + \dots + c_{i-1} \underbrace{(v_{i-1} \cdot v_i)}_{=0} + c_{i+1} \underbrace{(v_{i+1} \cdot v_i)}_{=0} + \dots + c_k \underbrace{(v_k \cdot v_i)}_{=0}$$

contradiction. $\square$

DEF 22.5: let $V \subset \mathbb{R}^n$

an **orthogonal basis** $\{v_1, \dots, v_k\}$ of V is a basis of V consisting of mutually orthogonal vectors



$\{v_1, v_2\}$ is an orthogonal basis of V

PROP 22.6: any subspace $V \subset \mathbb{R}^n$ has (infinitely many) orthogonal bases.

Proof: pick $0 \neq v_1 \in V$

let $v_1^\perp \cap V$

pick $0 \neq v_2$ in here

let $\text{span}\{v_1, v_2\}^\perp \cap V$

pick $0 \neq v_3$ in here

$\vdots$

$v_1, \ldots, v_k \in V$; by construction, each $v_i \in \text{span}\{v_1, \ldots, v_{i-1}\}^\perp$

$\Downarrow$

$v_i \perp v_j \quad \forall i \neq j$ $\square$

New topic: projections onto subspaces

Warm-up: subspace $V \subseteq \mathbb{R}^n$

orthogonal basis $\{v_1, \ldots, v_k\}$ of V

arbitrary $w \in V$

$\exists\, c_1, \ldots, c_k \in \mathbb{R}$ s.t. $w = c_1 v_1 + \cdots + c_k v_k$

how to calculate $c_1, \ldots, c_k$?

old way: $A = (v_1 | \cdots | v_k)$, solve $A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = w$

THM 22.7 new way for an orthogonal basis $\{v_1,\dots,v_k\}$ of V

$$c_1 = \frac{w\cdot v_1}{\|v_1\|^2}, \dots, c_k = \frac{w\cdot v_k}{\|v_k\|^2}$$

Proof: $w = c_1 v_1 + \dots + c_k v_k \quad \forall w \in V$

{ dot with any v_i

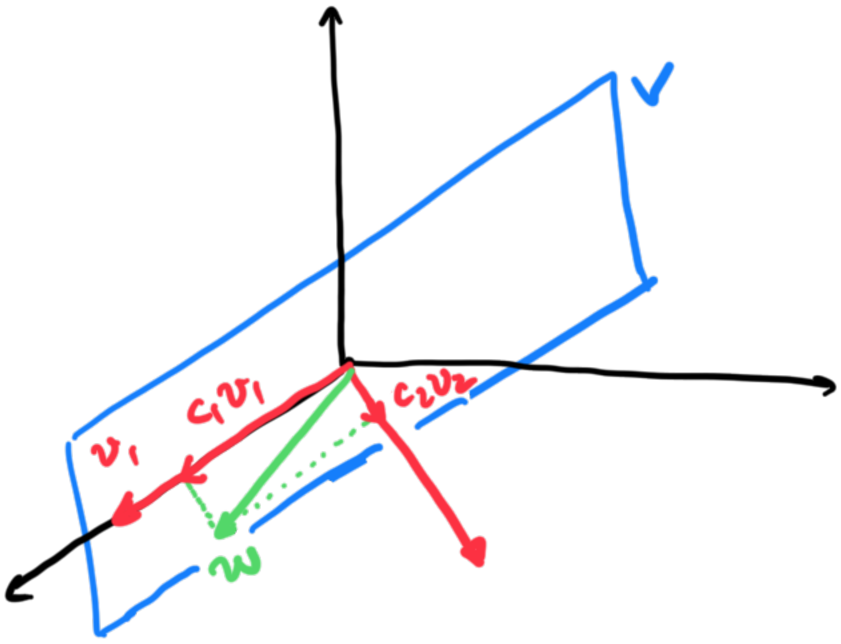
$$w\cdot v_i = c_1\underbrace{(v_1\cdot v_i)}_{=0} + \dots + c_{i-1}\underbrace{(v_{i-1}\cdot v_i)}_{=0} + c_i\underbrace{(v_i\cdot v_i)}_{\|v_i\|^2} + c_{i+1}\underbrace{(v_{i+1}\cdot v_i)}_{=0} + \dots + c_k\underbrace{(v_k\cdot v_i)}_{=0}$$

$\Downarrow$

$$w\cdot v_i = c_i\|v_i\|^2 \Rightarrow c_i = \frac{w\cdot v_i}{\|v_i\|^2}$$

Geometric interpretation:

$$c_1 = \frac{w\cdot v_1}{\|v_1\|^2}, \quad c_2 = \frac{w\cdot v_2}{\|v_2\|^2}$$



THM 22.8: let $V \subset \mathbb{R}^n$ be any subspace

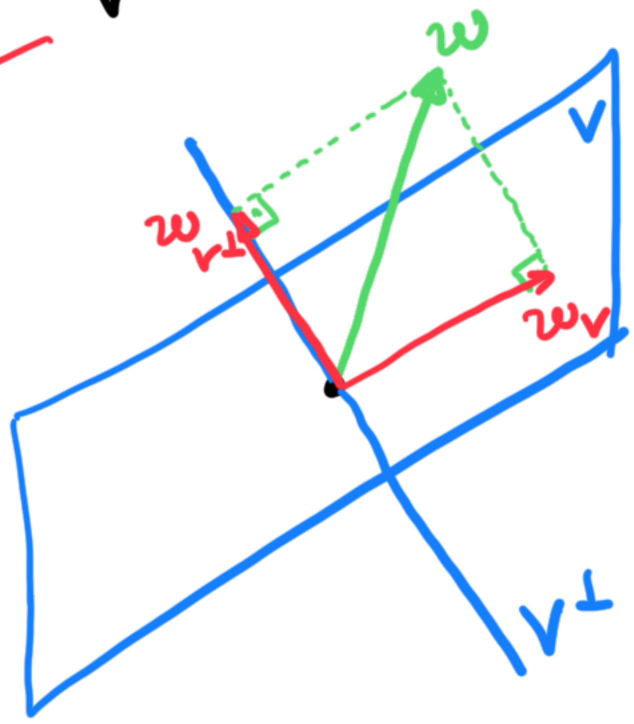
$\forall$ vector $w \in \mathbb{R}^n$, we can decompose it uniquely as

$$w = w_V + w_{V^\perp}$$

$V^\perp$

where $w_V \in V$ and $w_{V^\perp} \in V^\perp$

$\text{proj}_V(w)$ $\quad$ $\text{proj}_{V^\perp}(w)$



If you have an orthogonal basis $\{v_1, \dots, v_k\}$ of V

$$\text{proj}_V(w) = \frac{w \cdot v_1}{\|v_1\|^2} v_1 + \dots + \frac{w \cdot v_k}{\|v_k\|^2} v_k$$

coefficient in $\mathbb{R}$ $\quad$ coefficient in $\mathbb{R}$

Proof: let $c_1 = \frac{w \cdot v_1}{\|v_1\|^2}, \dots, c_k = \frac{w \cdot v_k}{\|v_k\|^2}$

define $y = c_1 v_1 + \dots + c_k v_k \in V$

y is a great candidate for $\text{proj}_V(w)$, because

$(w - y) \perp V$ $\quad$ by geometry

$\Updownarrow$

$(w-y) \perp v_i$ for all $i \in \{1, \dots, k\}$

$\Updownarrow$

$$0 = (w - y) \cdot v_i$$

recall that $y = c_1 v_1 + \ldots + c_k v_k = \frac{w \cdot v_1}{\|v_1\|^2} v_1 + \ldots + \frac{w \cdot v_k}{\|v_k\|^2} v_k$

$\Downarrow$ b/c $\{v_1, \ldots, v_k\}$ are orthogonal

$$y \cdot v_i = \frac{w \cdot v_i}{\cancel{\|v_i\|^2}} \cancel{v_i \cdot v_i} = w \cdot v_i$$

Ex: $\mathbb{R}^3 \supset V = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = v_1$, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = v_2$, $v_1 \cdot v_2 = 2 \cdot 0 + 1 \cdot 1 + 1 \cdot (-1) = 0$

$$? = \text{proj}_V (w \in \mathbb{R}^3) = \text{proj}_V \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Step 1: make sure you have an orthogonal basis of V

(if not, there's an algorithm called Gram-Schmidt that turns any basis into an orthogonal basis)

step 2: apply the formula

$$\text{proj}_V(w) = \frac{w \cdot v_1}{\|v_1\|^2} v_1 + \frac{w \cdot v_2}{\|v_2\|^2} v_2$$

$\| \qquad \| \qquad \|$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\left.\begin{aligned} w \cdot v_1 &= 2x+y+z \\ w \cdot v_2 &= y-z \\ \|v_1\|^2 &= 2^2+1^2+1^2=6 \\ \|v_2\|^2 &= 0^2+1^2+(-1)^2=2 \end{aligned}\right\} \qquad \mathrm{proj}_V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2x+y+z}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \frac{y-z}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

make sure RHS is linear in the coordinates of w, and that $\mathrm{proj}_V(0)=0$.

Note: the summands $\frac{w \cdot v_i}{\|v_i\|^2} v_i$ are unchanged by rescaling $v_1, \dots, v_k$ independently

Let's rescale $v_1, \dots, v_k$ such that they have length 1

DEF 22.9: an **orthonormal basis** is an orthogonal basis where all vectors have length 1

$\{v_1, \dots, v_k\}$

$\boxed{v_i \cdot v_i = 1}$ $\quad \|v_1\|^2 = \dots = \|v_k\|^2 = 1$

$v_i \cdot v_j = 0 \iff v_i \perp v_j \quad \forall i \neq j$

(Ex: $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$)

$\boxed{v_i \cdot v_j = 0}$ $\quad v_i \perp v_j, \ i \neq j$

(orthonormal bases do not need denominators in the formula for $\mathrm{proj}_V(w)$)

$\{v_1, \ldots, v_k\}$ an orthonormal basis $\Rightarrow A^T A = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_k \\ \vdots & & \\ v_k \cdot v_1 & \cdots & v_k \cdot v_k \end{pmatrix} = I_k$

$A = (v_1 | \cdots | v_k)$

DEF 22.10 : matrices $A \in \mathbb{R}^{n \times n}$ s.t. $A^T A = I_n$ are called **orthogonal**